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FULLNESS, CONNES' χ-GROUPS, AND ULTRA-PRODUCTS OF AMALGAMATED FREE PRODUCTS OVER CARTAN SUBALGEBRAS

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Abstract. Ultra-product algebras associated with amalgamated free products over Cartan subalgebras are investigated. As applications, their Connes' χ-groups are computed in terms of ergodic theory, and also we clarify what condition makes them full factors (i.e., their inner automorphism groups become closed).

1. Introduction

Let $A \supseteq D \subseteq B$ be von Neumann algebras and a common Cartan subalgebra. Then we can consider the amalgamated free product $M = A *_D B$ with respect to the unique conditional expectations. In the previous paper [25], the questions of factoriality and type classification of M were discussed in detail. Among other things, there exists a type III_0 example arising as the amalgamated free product of two non-type I factors over a common Cartan subalgebra.

On the other hand, many free products (or amalgamated free products over $\mathbf{C}$) of von Neumann algebras are known to be full in the sense of Connes [3] and not to be of type III_0 (see Barnett [1], Dykema [8]); and these two facts are related to each other because Connes [3] showed that no type III_0 factor is full. Hence, it might be worth investigating what kind of condition makes the amalgamated free product M in question a full factor. The main purpose of this paper is to give an answer to this question, and to compute the χ-group $\chi(M)$ in terms of ergodic theory. Here, it should be noticed that the χ-group of a certain amalgamated free product was considered by Rădulescu [20] for a different purpose.

This paper is organized as follows. In §2, we summarize basic definitions and properties and fix notation on Cartan subalgebras, ultra-products and Connes' χ-groups. In §3, a technical result on ultra-products of amalgamated free products is given, and based on it we will give, in §4, a necessary and sufficient condition for the amalgamated free product M to be full, and compute the χ-group $\chi(M)$ under the assumption that both A and B are factors not of type I. The final §5 is an appendix, where we show that the triple $A \supseteq D \subseteq B$ produces a unique(!) pair of equivalence relations over a common measure space, which is crucial throughout our analysis.

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2. Preliminaries

2.1. Cartan subalgebras. [[9]] Let $\mathcal{R}$ be a countable nonsingular Borel equivalence relation over a standard Borel probability space (X, μ), or, what is the same, the equivalence relation coming from a countable nonsingular transformation group on (X, μ) (see [9, I, Theorem 1]), and let σ be a 2-cocycle of $\mathcal{R}$. The right counting measure μ_r on $\mathcal{R}$ ($d\mu_r(x, y) = d\mu(y)$) associated with μ gives the Hilbert space $\mathcal{H} := L^2(\mathcal{R}, \mu_r)$. For a "nice" measurable function f on $\mathcal{R}$, we define the left convolution operator L^σ_f on $\mathcal{H}$ by

$$(L^\sigma_f \xi)(x, y) := \sum_{x\sim z\sim y} f(x, z)\xi(z, y)\sigma(x, z, y), \quad \xi \in \mathcal{H},$$

and denote by $W^*_\sigma(\mathcal{R})$ the von Neumann algebra generated by the L^σ_f's. It can be shown that any element in $W^*_\sigma(\mathcal{R})$ can be written as a convolution operator L^σ_f in an appropriate sense, and that the action of $W^*_\sigma(\mathcal{R})$ on $\mathcal{H}$ is standard. The L^∞-algebra $L^\infty(X)$ can be naturally embedded into $W^*_\sigma(\mathcal{R})$ as the subalgebra consisting of those L^σ_f with $\mathrm{supp}(f) \subseteq \Delta$, which is shown to be a MASA in $W^*_\sigma(\mathcal{R})$. Here, Δ means the diagonal set in $\mathcal{R}$. In what follows, we will freely identify a function $f \in L^\infty(X)$ with the corresponding operator in $W^*_\sigma(\mathcal{R})$ via the embedding. The mapping $L^\sigma_f \mapsto L^\sigma_{(\chi_\Delta\cdot f)}$ gives rise to the unique normal conditional expectation $E : W^*_\sigma(\mathcal{R}) \to L^\infty(X)$. It is plain to see that the normalizer $\mathcal{N}_{W^*_\sigma(\mathcal{R})}(L^\infty(X))$ generates the whole $W^*_\sigma(\mathcal{R})$. Therefore, $L^\infty(X)$ is a Cartan subalgebra in $W^*_\sigma(\mathcal{R})$. Conversely, any von Neumann algebra with a Cartan subalgebra arises in this way (see [9, II]).

Let

$$\mathrm{Gal}\left(W^*_\sigma(\mathcal{R}) \supseteq L^\infty(X)\right) := \{\alpha \in \mathrm{Aut}(W^*_\sigma(\mathcal{R}))\ ;\ \alpha|_{L^\infty(X)} = \mathrm{Id}\}$$

be the Galois group of the inclusion $W^*_\sigma(\mathcal{R}) \supseteq L^\infty(X)$. Let us choose an automorphism α from the Galois group, and denote its canonical unitary implementation on the standard form $\mathcal{H}$ (see e.g. [10]) by u_α. The unitary u_α is shown to be the multiplication operator m_c of a measurable function $c : \mathcal{R} \mapsto \mathbb{T}$ satisfying the 1-cocycle condition:

$$c(x, y)c(y, z) = c(x, z), \quad c(x, y) = \overline{c(y, x)},$$

and $\alpha = \mathrm{Ad}\, m_c|_{W^*_\sigma(\mathcal{R})}$ is nothing less than the multiplier $M^\sigma_c(L^\sigma_f) := L^\sigma_{c\cdot f}$ on $W^*_\sigma(\mathcal{R})$. In this way, we get the group isomorphism

$$Z^1(\mathcal{R}, \mathbb{T}) \cong \mathrm{Gal}\left(W^*_\sigma(\mathcal{R}) \supseteq L^\infty(X)\right); \quad c \leftrightarrow M^\sigma_c,$$

where $Z^1(\mathcal{R}, \mathbb{T})$ denotes the group of 1-cocycles of $\mathcal{R}$ taking values in $\mathbb{T}$ (regarded as a subset of $L^\infty(\mathcal{R}, \mu_r)$).

Let M be a von Neumann algebra with separable predual M_*. The most natural topology on the automorphism group $\mathrm{Aut}(M)$ is the so-called u-topology, i.e.,

$$u\text{-}\lim_{n\to\infty} \alpha_n = \alpha \Longleftrightarrow \lim_{n\to\infty} ||\psi \circ \alpha_n - \psi \circ \alpha|| = 0 \text{ for every } \psi \in M_*.$$

The u-topology makes $\mathrm{Aut}(M)$ a complete metrizable group, and the canonical implementation

$$\alpha \in \mathrm{Aut}(M) \longmapsto u_\alpha \in \mathcal{U}\left(L^2(M)\right)$$

gives a bi-continuous injective homomorphism when the unitary group $\mathcal{U}\left(L^2(M)\right)$ is equipped with the strong operator topology. Through the correspondences

$$c \in Z^1(\mathcal{R}, \mathbb{T}) \longleftrightarrow M_c^\sigma \in \mathrm{Gal}\left(W^*(\mathcal{R}) \supseteq L^\infty(X)\right) \longleftrightarrow m_c \in \mathcal{U}(\mathcal{H}),$$

we can easily see the following:

Proposition 1 ([9])**.** *The group isomorphism*

$$c \in Z^1(\mathcal{R}, \mathbb{T}) \longleftrightarrow M_c^\sigma \in \mathrm{Gal}\left(W_\sigma^*(\mathcal{R}) \supseteq L^\infty(X)\right) \tag{1}$$

is a homeomorphism with respect to the topology of convergence in probability (*equivalent to the right counting measure* μ_r) *and the u-topology, respectively.*

A 1-cocycle c of $\mathcal{R}$ is a coboundary if there is a measurable function $u : X \to \mathbb{T}$ such that

$$c(x, y) = u(x)u(y)^{-1} \ (= u(x)\overline{u(y)}),$$

and the subset of coboundaries is denoted by $B^1(\mathcal{R}, \mathbb{T})$. It is plain to see that a coboundary $c(x, y) = u(x)u(y)^{-1}$ satisfies

$$M_c^\sigma(L_f^\sigma) = L_{c\cdot f}^\sigma = \mathrm{Ad}\, u\left(L_f^\sigma\right),$$

and $B^1(\mathcal{R}, \mathbb{T})$ corresponds to $\mathrm{Int}(W_\sigma^*(\mathcal{R}), L^\infty(X))$ via (1). Here,

$$\mathrm{Int}(W_\sigma^*(\mathcal{R}), L^\infty(X)) := \{\mathrm{Ad}\, u \in \mathrm{Int}(W_\sigma^*(\mathcal{R}))\ ;\ u \in L^\infty(X)\}.$$

The first cohomology group is defined as follows:

$$H^1(\mathcal{R}, \mathbb{T}) := Z^1(\mathcal{R}, \mathbb{T})/B^1(\mathcal{R}, \mathbb{T}).$$

2.2. Ultra-products of von Neumann algebras. [[14], [3], [15], [18]] Let M be a von Neumann algebra with separable predual M_*, and fix a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$. We set

- $\mathcal{I}_\omega^M := \left\{(m(k))_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, M)\ ;\ \sigma\text{-}s^*\text{-}\lim_{k\to\omega} m(k) = 0\right\}$,
- $\mathcal{M}(\mathcal{I}_\omega^M) :=$ the multiplier algebra of $\mathcal{I}_\omega^M$ in $\ell^\infty(\mathbb{N}, M)$,
- $\mathcal{C}_\omega(M) := \left\{(m(k))_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, M)\ ;\ \lim_{k\to\omega} ||[m(k), \psi]|| = 0,\ \psi \in M_*\right\}$.

Then we see that

$$\mathcal{M}(\mathcal{I}_\omega^M) \supseteq \mathcal{C}_\omega(M) \supseteq \mathcal{I}_\omega^M, \quad M \hookrightarrow \mathcal{M}(\mathcal{I}_\omega^M) \text{ by } m \mapsto (m, m, \dots).$$

The quotient C^*-algebra $\mathcal{M}(\mathcal{I}_\omega^M)/\mathcal{I}_\omega^M$ is called the *ultra-product* of M at ω and denoted by M^ω ([15, Chap. 5]). The canonical quotient map is denoted by $\pi : \mathcal{M}(\mathcal{I}_\omega^M) \to \mathcal{M}(\mathcal{I}_\omega^M)/\mathcal{I}_\omega^M = M^\omega$. The C^*-subalgebra $\pi(\mathcal{C}_\omega(M)) = \mathcal{C}_\omega(M)/\mathcal{I}_\omega^M$ is called the *asymptotic centralizer* of M at ω, and is denoted by M_ω ([3]). The original M can be identified with $\pi(M) = (M + \mathcal{I}_\omega^M)/\mathcal{I}_\omega^M$. In [15, Proposition 5.1], it was shown that (i) M^ω is a von Neumann algebra, and both M_ω and $M = \pi(M)$ are its von Neumann subalgebras; (ii) for each faithful positive $\phi \in M_*$, the linear functional $\phi^\omega : M^\omega \to \mathbf{C}$ defined by

$$\phi^\omega(\pi((m(k))_{k\in\mathbb{N}})) := \lim_{k\to\omega} \phi(m(k)), \quad (m(k))_{k\in\mathbb{N}} \in \mathcal{M}(\mathcal{I}_\omega^M),$$

is faithful normal.

It is known that $M' \cap M_\omega = M_\omega$, and hence $M_\omega \subseteq M' \cap M^\omega$. Indeed, this follows from the fact that, for each $m_1, m_2 \in M$,

$$||[m_1, m_2]||_\varphi \leq 4 \cdot \max\{|||[\varphi, m_2]|||,\ |||[m_1\varphi, m_2]|||,\ |||[m_1\varphi, m_2^*]|||\}.$$

See [3, p. 425]. However, the reverse inclusion relation $M' \cap M^\omega \subseteq M_\omega$ does not hold in general, as was pointed out by Barnett and Takesaki (see [18, p. 22]).

Let N be a von Neumann subalgebra of M and $N^\omega = \mathcal{M}(\mathcal{I}_\omega^N)/\mathcal{I}_\omega^N$. Suppose that there is a faithful normal conditional expectation $E : M \to N$, and further that the above φ is chosen in such a way that $\varphi \circ E = \varphi$. Then it is easy to check that

$$(n(k))_{k\in\mathbb{N}} \in \mathcal{M}(\mathcal{I}_\omega^N) \Longrightarrow (n(k))_{k\in\mathbb{N}} \in \mathcal{M}(\mathcal{I}_\omega^M).$$

Thanks to this implication, there is a natural embedding of N^ω into M^ω, and thus we will regard N^ω as a subalgebra of M^ω. It can be shown that N^ω is a von Neumann subalgebra as well, which indeed follows from the fact that the norm $||\ \cdot\ ||_{(\varphi|_N)^\omega}$ on N^ω coincides with the restriction of $||\ \cdot\ ||_{\varphi^\omega}$ to N^ω. It is also easy to check that

$$(m(k))_{k\in\mathbb{N}} \in \mathcal{I}_\omega^M \Longrightarrow (E(m(k)))_{k\in\mathbb{N}} \in \mathcal{I}_\omega^N,$$
$$(m(k))_{k\in\mathbb{N}} \in \mathcal{M}(\mathcal{I}_\omega^M) \Longrightarrow (E(m(k)))_{k\in\mathbb{N}} \in \mathcal{M}(\mathcal{I}_\omega^N),$$

and hence the equation

$$E^\omega(\pi((m(k))_{k\in\mathbb{N}})) := \pi((E(m(k)))_{k\in\mathbb{N}})$$

gives rise to a well-defined conditional expectation $E^\omega : M^\omega \to N^\omega$. The construction guarantees that $\varphi^\omega \circ E^\omega = \varphi^\omega$ thanks to $\varphi \circ E = \varphi$, and hence E^ω is normal.

Let $\mathcal{H}_{\varphi^\omega}$ be the standard Hilbert space associated with $(M^\omega, \varphi^\omega)$, and denote by $\Lambda_{\varphi^\omega} : M^\omega \to \mathcal{H}_{\varphi^\omega}$ the canonical injection. We have

$$(2) \qquad \left(\Lambda_{\varphi^\omega}(\pi((m_1(k))_{k\in\mathbb{N}}) \middle| \Lambda_{\varphi^\omega}(\pi((m_2(k))_{k\in\mathbb{N}}))\right)_{\varphi^\omega} = \lim_{k\to\omega}(m_1(k)|m_2(k))_\varphi$$

for $(m_1(k))_{k\in\mathbb{N}}, (m_2(k))_{k\in\mathbb{N}} \in \mathcal{M}(\mathcal{I}_\omega^M)$. Let $\mathcal{H}_\varphi$ be the standard Hilbert space associated with (M, φ), and we will consider its ultra-product at ω into which the Hilbert space $\mathcal{H}_{\varphi^\omega}$ can be embedded. Set

$$\mathcal{N}_\varphi^\omega := \left\{(\xi(k))_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathcal{H}_\varphi)\ ;\ \lim_{k\to\omega}||\xi(k)||_\varphi = 0\right\},$$

and define

$$\mathcal{H}_\varphi^\omega := \ell^\infty(\mathbb{N}, \mathcal{H}_\varphi)/\mathcal{N}_\varphi^\omega$$

equipped with the inner product

$$(3) \qquad ([(\xi(k))_{k\in\mathbb{N}}]\ |\ [(\zeta(k))_{k\in\mathbb{N}}])_\varphi^\omega := \lim_{k\to\omega}(\xi(k)|\zeta(k))_\varphi,$$

where $[(\xi(k))_{k\in\mathbb{N}}]$, $[(\zeta(k))_{k\in\mathbb{N}}]$ denote the equivalence classes of $(\xi(k))_{k\in\mathbb{N}}$, $(\zeta(k))_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathcal{H}_\varphi)$. One can show that $\mathcal{H}_\varphi^\omega$ is complete with respect to the inner product, and it is called the ultra-product of $\mathcal{H}_\varphi$ at ω. It is clear from (2), (3) that $\mathcal{H}_{\varphi^\omega}$ is embedded into $\mathcal{H}_\varphi^\omega$ via the mapping

$$\Lambda_{\varphi^\omega}(\pi((m(k))_{k\in\mathbb{N}})) \in \mathcal{H}_{\varphi^\omega} \longmapsto [(\Lambda_\varphi(m(k)))_{k\in\mathbb{N}}] \in \mathcal{H}_\varphi^\omega.$$

2.3. Connes' χ-groups. [[5], [6]] Let M be a von Neumann algebra with separable predual M_*. The p-topology on $\mathrm{Aut}(M)$ is defined as follows:

$$p\text{-}\lim_{n\to\infty} \alpha_n = \alpha \Longleftrightarrow \lim_{n\to\infty} ||\alpha_n(x) - \alpha(x)||_\varphi = 0 \quad \text{for every } x \in M,$$

where φ is a fixed faithful normal state on M. We set

$$\mathrm{Aut}_\varphi(M) := \{\alpha \in \mathrm{Aut}(M) \ ; \ \varphi \circ \alpha = \varphi\},$$

a subgroup of $\mathrm{Aut}(M)$. The u-topology is stronger than the p-topology. However, it is known (see [10, Proposition 3.7]) that the p-topology coincides with the u-topology on $\mathrm{Aut}_\varphi(M)$. This fact will be used later, repeatedly and crucially.

A centralizing sequence $(m(k))_{k\in\mathbb{N}}$ in M is a bounded sequence of elements of M with $\lim_{k\to\infty} ||[m(k), \psi]|| = 0$ for every $\psi \in M_*$. An automorphism $\alpha \in \mathrm{Aut}(M)$ is said to be centrally trivial if $\sigma\text{-}s^*\text{-}\lim_{k\to\infty} (\alpha(m(k)) - m(k)) = 0$ for every centralizing sequence $(m(k))_{k\in\mathbb{N}}$ in M. The set of centrally trivial automorphisms forms a normal subgroup of $\mathrm{Aut}(M)$, denoted by $\mathrm{Ct}(M)$. For every free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$, we know that $M_\omega \subseteq M' \cap M^\omega$ (see §2.2), and hence $\mathrm{Int}(M)$ sits in $\mathrm{Ct}(M)$.

Definition ([5])**.** We set

$$\chi(M) := \text{the center}\left(\overline{\mathrm{Int}(M)}/\mathrm{Int}(M)\right) = \frac{\mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)}}{\mathrm{Int}(M)},$$

and call it the χ-group of M. Here, $\overline{\mathrm{Int}(M)}$ means the closure of $\mathrm{Int}(M)$ with respect to the u-topology.

Note that the second equality comes from [4, Corollary 2.3.2]. However, we will deal with χ-groups without any use of this general result of Connes.

The following proposition is an important tool for our computation of the χ-group in the next section. It is believed to be a folklore result for specialists, and indeed it follows from a straightforward adaptation of Connes' method ([6]) to our slightly generalized situation.

Proposition 2 (See [6, Theorem 2.1])**.** *Let D be a finite von Neumann subalgebra of M with a faithful normal conditional expectation $E_D^M : M \to D$. If every centralizing sequence $(m(k))_{k\in\mathbb{N}}$ in M satisfies*

$$\sigma\text{-}s^*\text{-}\lim_{k\to\infty} \left(m(k) - E_D^M(m(k))\right) = 0,$$

then, for each $\beta \in \overline{\mathrm{Int}(M)}$, one can choose a unitary $X \in M$ and an automorphism $\beta_0 \in \overline{\mathrm{Int}(M, D)}$ in such a way that

$$\beta = (\mathrm{Ad}\, X) \circ \beta_0,$$

where $\mathrm{Int}(M, D) := \{\mathrm{Ad}\, u \in \mathrm{Aut}(M) : u \in \mathcal{U}(D)\}$.

The finiteness condition of the subalgebra D is crucial; see the proof of [6, Theorem 2.1].

3. Amalgamated free products and their ultra-products

Let $A \supseteq D \subseteq B$ be von Neumann algebras with separable preduals, and suppose that there are faithful normal conditional expectations $E_D^A : A \to D$, $E_D^B : B \to D$. The amalgamated free product

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B)$$

is a pair consisting of a von Neumann algebra M into which the triple $A \supseteq D \subseteq B$ is embedded and a faithful normal conditional expectation $E_D^M : M \to D$, and characterized by the following three conditions:

- M is generated by the subalgebras A, B;
- $E_D^M|_A = E_D^A$ and $E_D^M|_B = E_D^B$;
- A, B are free with amalgamation over D in the D-probability space $(M \supseteq D, E_D^M)$, i.e.,

  $$E_D^M(\{\text{alternating words in } A^\circ, B^\circ\}) = 0,$$

  where we denote $A^\circ := \mathrm{Ker}E_D^A$, $B^\circ := \mathrm{Ker}E_D^B$ as usual.

For the details, we refer to [17], [25], [27] (also [2]).

Proposition 3 ([25, Theorem 2.6])**.** *Let φ be a faithful normal state on D. The modular automorphism $\sigma_t^{\varphi\circ E_D^M}$ satisfies*

$$\sigma_t^{\varphi\circ E_D^M}|_A = \sigma_t^{\varphi\circ E_D^A}, \quad \sigma_t^{\varphi\circ E_D^M}|_B = \sigma_t^{\varphi\circ E_D^B}.$$

Fix an arbitrary free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Let $M^\omega \supseteq A^\omega, B^\omega \supseteq D^\omega$ be the ultra-products at ω. Note here that there are faithful normal conditional expectations from M onto A, B thanks to Proposition 3. Let $(E_D^M)^\omega : M^\omega \to D^\omega$, $(E_D^A)^\omega : A^\omega \to D^\omega$, $(E_D^B)^\omega : B^\omega \to D^\omega$ be the natural liftings of E_D^M, E_D^A, E_D^B, respectively. (See §3.)

Proposition 4. *The von Neumann subalgebras A^ω and B^ω are free with amalgamation over D^ω in the D^ω-probability space $(M^\omega \supseteq D^\omega, (E_D^M)^\omega)$.*

Proof. Let us choose an alternating word $x = x_1^\circ \cdots x_n^\circ$ in $(A^\omega)^\circ$, $(B^\omega)^\circ$. Assume for a while that x_j° is in $(A^\omega)^\circ$. Let $(x_j(k))_{k\in\mathbb{N}}$ be a representative of x_j°. We have

$$\begin{aligned} x_j^\circ &= x_j^\circ - (E_D^A)^\omega\left(x_j^\circ\right) \\ &= \pi\left((x_j(k))_{k\in\mathbb{N}}\right) - \pi\left(\left(E_D^A(x_j(k))\right)_{k\in\mathbb{N}}\right) \\ &= \pi\left(\left(x_j(k) - E_D^A(x_j(k))\right)_{k\in\mathbb{N}}\right), \end{aligned}$$

so that every $x_j(k)$ can be replaced by an element in A°. Hence we may and do assume that our representatives of the x_j°'s are chosen as above, i.e., $(x_j(k)^\circ)_{k\in\mathbb{N}}$ with $E_D^A(x_j(k)^\circ) = 0$ or $E_D^B(x_j(k)^\circ) = 0$. Therefore, we have

$$(E_D^M)^\omega(x_1^\circ \cdots x_n^\circ) = \lim_{k\to\omega} E_D^M(x_1(k)^\circ \cdots x_n(k)^\circ) = 0$$

by the freeness of A, B in the D-probability space $(M \supseteq D, E_D^M)$. $\square$

One might expect that it would follow from the proposition that the ultra-product algebra M^ω can be written as the amalgamated free product of A^ω and B^ω over D^ω.

However, this may or may not be true. In fact, it is highly nontrivial whether or not the subalgebras A^ω, B^ω generate M^ω.

In the proposition below, we use two different Hilbert spaces. One is the GNS-Hilbert space associated with the ultra-product algebra M^ω and the other is that associated with the original algebra M. Thus we would like to use the following notation rule to avoid any confusion:

$L^2(N,\phi)$, the GNS-Hilbert space associated with (N,ϕ),
$\Lambda_\phi : N \to L^2(N,\phi)$, the canonical injection,

for a pair (N,ϕ) consisting of a von Neumann algebra and a faithful normal state.

Proposition 5 (cf. [16, Lemma 2.1])**.** *Suppose there are a faithful normal state φ on D and two unitaries $u, w \in \mathcal{U}(A_{\varphi\circ E_D^A})$ such that $uDu^* = D = wDw^*$ (automatically satisfied by the fact they sit in the centralizer) and*

$$E_D^A(u^n) = 0, \quad E_D^A(w^n) = 0$$

as long as $n \neq 0$. Then, for any $x \in M^\omega$ with $x = uxw^*$ and for any pair $y_1, y_2 \in B^\circ$, we have

$$\left|\left|\Lambda_{(\varphi\circ E_D^M)^\omega}(y_1x - xy_2)\right|\right|^2_{(\varphi\circ E_D^M)^\omega} \geq \left|\left|\Lambda_{(\varphi\circ E_D^M)^\omega}(y_1(x - (E_D^M)^\omega(x)))\right|\right|^2_{(\varphi\circ E_D^M)^\omega} + \left|\left|\Lambda_{(\varphi\circ E_D^M)^\omega}((x - (E_D^M)^\omega(x))y_2)\right|\right|^2_{(\varphi\circ E_D^M)^\omega}.$$

Proof. Since the embedding of M into M^ω is normal, we may and do assume that y_1, y_2 are analytic elements for the modular action $\sigma^{\varphi\circ E_D^M}$ (or equivalently for $\sigma^{\varphi\circ E_D^B}$, thanks to Proposition 3). Note here that the analytic elements form a σ-strong* dense $*$-subalgebra, and that, if y is analytic, then so is the new element $y^\circ := y - E_D^B(y)$.

We define the following subspaces in $L^2(M, \varphi\circ E_D^M)$:

$$\begin{aligned}
\mathcal{X}_1 &:= \text{the closed subspace generated by } \Lambda_{\varphi\circ E_D^M}(A^\circ \cdots B^\circ);\\
\mathcal{X}_2 &:= \text{the closed subspace generated by } \Lambda_{\varphi\circ E_D^M}(B^\circ \cdots A^\circ);\\
\mathcal{X}_3 &:= \text{the closed subspace generated by } \Lambda_{\varphi\circ E_D^M}(B^\circ \cdots B^\circ);\\
\mathcal{X}_4 &:= \text{the closed subspace generated by } \Lambda_{\varphi\circ E_D^M}(A^\circ \cdots A^\circ).
\end{aligned}$$

Then we see that

$$L^2(M, \varphi\circ E_D^M) = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \oplus \overline{\Lambda_{\varphi\circ E_D^M}(D)}.$$

We also introduce the operator $T_{(u^n,w^n)}$ $(n \in \mathbb{Z})$ on $L^2(M, \varphi\circ E_D^M)$ defined by

$$T_{(u^n,w^n)}\Lambda_{\varphi\circ E_D^M}(x) := \Lambda_{\varphi\circ E_D^M}(u^n x w^{-n}), \quad x \in M.$$

Since both u and w are in the centralizer $M_{\varphi\circ E_D^M}$ (thanks to Proposition 3), one can check that

- $T_{(u^n,w^n)}$ is a unitary;
- $T_{(u,w)}^n = T_{(u^n,w^n)}$, $n \in \mathbb{Z}$;
- $T_{(u,w)}^n P_{\mathcal{X}_i} = P_{(T_{(u,w)}^n \mathcal{X}_i)} T_u^n$,

where $P_{\mathcal{Y}}$ denotes the projection onto a closed subspace $\mathcal{Y}$.

Claim. *For $i = 1, 2, 3$ $(\neq 4)$, we have $T^n_{(u,w)}\mathcal{X}_i \perp T^m_{(u,w)}\mathcal{X}_i$ as long as $n \neq m$.*

Proof of the Claim. Notice first that both u^n and w^n are in A° as long as $n \neq 0$. Thus, the case of $\mathcal{X}_3$ is trivial, and also the case of $\mathcal{X}_2$ is relatively easy (see the proof of [25, Proposition 4.1]). Therefore, we here discuss only the case of $\mathcal{X}_1$, and the other cases are left to the reader.

For two alternating words $x_1 = a_1^\circ \cdots b_1^\circ$, $x_2 = a_2^\circ \cdots b_2^\circ$ in A° and B°, we have

$$\begin{aligned}
&\left(T^n_{(u,w)}\Lambda_{\varphi\circ E_D^M}(x_1)\middle|\, T^m_{(u,w)}\Lambda_{\varphi\circ E_D^M}(x_2)\right)_{\varphi\circ E_D^M} \\
&\qquad= \left(T^{n-m}_{(u,w)}\Lambda_{\varphi\circ E_D^M}(x_1)\middle|\, \Lambda_{\varphi\circ E_D^M}(x_2)\right)_{\varphi\circ E_D^M} \\
&\qquad= \left(T_{(u^{n-m},w^{n-m})}\Lambda_{\varphi\circ E_D^M}(x_1)\middle|\, \Lambda_{\varphi\circ E_D^M}(x_2)\right)_{\varphi\circ E_D^M} \\
&\qquad= \varphi\circ E_D^M\left(x_2^* u^{n-m} x_1 w^{m-n}\right) \\
&\qquad= \varphi\circ E_D^M\left(b_2^{\circ *}\cdots a_1^{\circ *} u^{n-m} a_1^\circ \cdots b_1^\circ w^{m-n}\right) \\
&\qquad= \varphi\circ E_D^A\left(E_D^B\left(b_2^{\circ *}\left(\cdots E_D^A\left(a_2^{\circ *} u^{n-m} a_1^\circ\right)\cdots\right) b_1^\circ\right) w^{m-n}\right).
\end{aligned}$$

If $m \neq n$, then the above value is zero since Dw^{m-n} is contained in the kernel of E_D^A. Hence we are done. $\square$

Let us return to the proof of theorem. Let $x = \pi((x(k))_{k\in\mathbb{N}})$ be an element satisfying $x = uxw^*$. Then for each fixed $n \in \mathbb{Z}$, we have

$$\begin{aligned}
&\lim_{k\to\omega}\left|\left|\Lambda_{\varphi\circ E_D^M}(x(k) - u^n x(k) w^{-n})\right|\right|_{\varphi\circ E_D^M} \\
&\qquad= \left|\left|\Lambda_{(\varphi\circ E_D^M)^\omega}(x - u^n x w^{-n})\right|\right|_{(\varphi\circ E_D^M)^\omega} = 0.
\end{aligned}$$

Thus, for each $\varepsilon > 0$ and for each $n_0 \in \mathbb{N}$, there is a neighborhood W at ω (with respect to the w^*-topology on $\beta(\mathbb{N})$) such that

$$\left|\left|\Lambda_{\varphi\circ E_D^M}(x(k) - u^n x(k) w^{-n})\right|\right|_{\varphi\circ E_D^M} < \varepsilon$$

for every $|n| \leq n_0$, $k \in W \cap \mathbb{N}$. In what follows, we write $T = T_{(u,w)}$ for short. For each $i \neq 4$ and for every $k \in W \cap \mathbb{N}$, $|n| \leq n_0$ (W and n_0 as above), we have

$$
\begin{aligned}
&\left|\left|P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&= \left|\left|T^n P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&= \left|\left|T^n P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k)) - P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k)) + P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&\le 2\left|\left|T^n P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k)) - P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&\qquad\qquad\qquad\qquad + 2\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&= 2\left|\left|P_{T^n\mathcal{X}_i}T^n\Lambda_{\varphi\circ E_D^M}(x(k)) - P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&\qquad\qquad\qquad\qquad + 2\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&= 2\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(u^n x(k) w^{-n} - x(k))\right|\right|_{\varphi\circ E_D^M}^2 + 2\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&\le 2\left|\left|\Lambda_{\varphi\circ E_D^M}(x(k) - u^n x(k) w^{-n})\right|\right|_{\varphi\circ E_D^M}^2 + 2\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&< 2\varepsilon^2 + 2\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
(2n_0+1)\left|\left|P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 &= \sum_{|n|\le n_0}\left|\left|P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&< \sum_{|n|\le n_0}\left\{2\varepsilon^2 + 2\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2\right\} \\
&= 2(2n_0+1)\varepsilon^2 + 2\sum_{|n|\le n_0}\left|\left|P_{T^n\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \\
&\le 2(2n_0+1)\varepsilon^2 + 2\left|\left|\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 \quad \text{(thanks to the Claim)} \\
&\le 2\left((2n_0+1)\varepsilon^2 + \left|\left|\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2\right) \\
&\le 2\left((2n_0+1)\varepsilon^2 + ||(x(k))_{k\in\mathbb{N}}||_\infty^2\right).
\end{aligned}
$$

Therefore, we have

$$
\left|\left|P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 < 2\left(\varepsilon^2 + \frac{1}{2n_0+1}||(x(k))_{k\in\mathbb{N}}||_\infty^2\right)
$$

as long as $k \in W \cap \mathbb{N}$. As a consequence, for each $\varepsilon > 0$, there is a neighborhood W_ε at ω such that

$$
\left|\left|P_{\mathcal{X}}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 = \sum_{i=1}^3\left|\left|P_{\mathcal{X}_i}\Lambda_{\varphi\circ E_D^M}(x(k))\right|\right|_{\varphi\circ E_D^M}^2 < \varepsilon^2 \tag{4}
$$

as long as $k \in W_\varepsilon \cap \mathbb{N}$, where $\mathcal{X} := \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$.

From now on, we regard $L^2(M^\omega, (\varphi\circ E_D^M)^\omega)$ as a (closed) subspace of the ultraproduct Hilbert space $\mathcal{H}^\omega := L^2(M, \varphi\circ E_D^M)^\omega$. We have

$$\begin{aligned}
&\left\|\Lambda_{(\varphi\circ E_D^M)^\omega}\left(y_1\left(x-(E_D^M)^\omega(x)\right)\right) - \left[\left(y_1 P_{\mathcal{X}_4}\Lambda_{\varphi\circ E_D^M}(x(k))\right)_{k\in\mathbb{N}}\right]\right\|_{\mathcal{H}^\omega}\\
&= \lim_{k\to\omega}\left\|\Lambda_{\varphi\circ E_D^M}\left(y_1\left(x(k)-E_D^M(x(k))\right)\right) - \left(y_1 P_{\mathcal{X}_4}\Lambda_{\varphi\circ E_D}(x(k))\right)\right\|_{\varphi\circ E_D^M}\\
&\le \sup_{k\in W_\varepsilon\cap\mathbb{N}}\left\|\Lambda_{\varphi\circ E_D^M}\left(y_1(x(k)-E_D^M(x(k))\right) - y_1 P_{\mathcal{X}_4}\Lambda_{\varphi\circ E_D}(x(k))\right\|_{\varphi\circ E_D^M}\\
&= \sup_{k\in W_\varepsilon\cap\mathbb{N}}\|y_1\|\cdot\left\|P_{\mathcal{X}}\Lambda_{\varphi\circ E_D^M}(x(k))\right\|_{\varphi\circ E_D^M}\\
&< \|y_1\|\cdot\varepsilon \qquad \text{(thanks to (4))},
\end{aligned}$$

and hence

$$\Lambda_{(\varphi\circ E_D^M)^\omega}\left(y_1(x-(E_D^M)^\omega(x)\right) = \left[\left(y_1 P_{\mathcal{X}_4}\Lambda_{\varphi\circ E_D^M}(x(k))\right)_{k\in\mathbb{N}}\right] \quad (\text{in } \mathcal{H}^\omega) \tag{5}$$

since ε is arbitrary. Since y_2 is an analytic element for $\sigma := \sigma^{\varphi\circ E_D^M}$, we have

$$\begin{aligned}
(\varphi\circ E_D^M)^\omega\left(y_2\pi\left((m(k))_{k\in\mathbb{N}}\right)\right) &= \lim_{k\to\omega}\varphi\circ E_D^M(y_2 m(k))\\
&= \lim_{k\to\omega}\varphi\circ E_D^M(m(k)\sigma_{-i}(y_2))\\
&= (\varphi\circ E_D^M)^\omega\left(\pi\left((m(k))_k\right)_{k\in\mathbb{N}}\sigma_{-i}(y_2)\right)\right)
\end{aligned}$$

for every $\pi((m(k))_{k\in\mathbb{N}}) \in M^\omega$. From this, in the same way as above, we have

$$\Lambda_{(\varphi\circ E_D^M)^\omega}\left((x-(E_D^M)^\omega(x)y_2\right) = \left[\left(J\sigma_{-\frac{i}{2}}(y_2^*)JP_{\mathcal{X}_4}\Lambda_{\varphi\circ E_D^M}(x(k))\right)_{k\in\mathbb{N}}\right], \tag{6}$$

where J is the modular conjugation associated with M. Also, we obtain

$$\begin{aligned}
&\Lambda_{(\varphi\circ E_D^M)^\omega}\left(y_1(E_D^M)^\omega(x) - (E_A^M)^\omega(x)y_2\right)\\
&\qquad = \left[\left(\Lambda_{\varphi\circ E_D^M}\left(y_1 E_A^M(x(k)) - E_D^M(x(k))y_2\right)\right)_{k\in\mathbb{N}}\right].
\end{aligned} \tag{7}$$

The right-hand sides of (5), (6), (7) are mutually orthogonal in $\mathcal{H}^\omega$, since so are

$$y_1(A^\circ\cdots A^\circ), \quad (A^\circ\cdots A^\circ)y_2, \quad y_1 D + D y_2\ (\subseteq B^\circ)$$

with respect to $\varphi\circ E_D^M$ thanks to the freeness of A, B. Therefore, we have

$$\begin{aligned}
&\left\|\Lambda_{(\varphi\circ E_D^M)^\omega}(y_1 x - x y_2)\right\|^2_{(\varphi\circ E_D^M)^\omega}\\
&\qquad = \left\|\Lambda_{(\varphi\circ E_D^M)^\omega}\left(y_1\left(x-(E_D^M)^\omega(x)\right)\right)\right\|^2_{(\varphi\circ E_D^M)^\omega}\\
&\qquad\quad + \left\|\Lambda_{(\varphi\circ E_D^M)^\omega}\left(\left(x-(E_D^M)^\omega(x)\right)y_2\right)\right\|^2_{(\varphi\circ E_D^M)^\omega}\\
&\qquad\qquad + \left\|\Lambda_{(\varphi\circ E_D^M)^\omega}\left(y_1(E_D^M)^\omega(x) - (E_D^M)^\omega(x)y_2\right)\right\|^2_{(\varphi\circ E_D^M)^\omega},
\end{aligned}$$

and hence we get the desired inequality. $\square$

4. Main results

Assume that both A and B are non-type I factors with separable preduals and that D is a common Cartan subalgebra. Thanks to [9, II, Theorem 1] (see also §5) there are two unique pairs $(\mathcal{R}_A,\,[\sigma_A])$ and $(\mathcal{R}_B,\,[\sigma_B])$ of ergodic countable nonsingular equivalence relations over a common non-atomic standard Borel probability space (X,μ) together with cohomology classes of 2-cocycles such that

$$A = W^*_{\sigma_A}(\mathcal{R}_A), \quad B = W^*_{\sigma_B}(\mathcal{R}_B), \quad \text{and} \quad D = L^\infty(X).$$

Let

$$(M, E^M_D) = (A, E^A_D) *_D (B, E^B_D)$$

be the amalgamated free product, where E^A_D and E^B_D are the unique conditional expectations, and we will write $M = A *_D B$ since no confusion is possible. We call

$$\mathcal{R}_M := \mathcal{R}_A \vee \mathcal{R}_B \quad (\subseteq X^2)$$

the *canonical equivalence relation* associated with $M = A*_D B$. We should note that D may or may not be a Cartan subalgebra of M. In fact, if $A = B = \langle D, u\rangle$, i.e., both are the same hyperfinite factor, then the subalgebra D can never be a MASA in M, since $u_A u_B^*$ commutes with D (since $\mathrm{Ad}\, u_A$ and $\mathrm{Ad}\, u_B$ both induce the same action on D) but is not contained in D, where u_A, u_B mean the corresponding copies of u in A, B (sitting in M), respectively. (More is true. Namely the relative commutant $D' \cap M$ becomes huge in this case.) Therefore, one cannot recover $\mathcal{R}_M$ from the pair $M \supseteq D$ (via the Feldman-Moore theorem [9, II]) without looking at $A \supseteq D \subseteq B$; so we need to show that $\mathcal{R}_M$ is really *canonical* for the triple $A \supseteq D \subseteq B$, which will be done in the Appendix.

Fix an arbitrary free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$. We will consider, as in §3, the ultra-product M^ω into which the ultra-products $A^\omega \supseteq D^\omega \subseteq B^\omega$ are embedded, and the natural liftings $(E^M_D)^\omega : M^\omega \to D^\omega$, $(E^A_D)^\omega : A^\omega \to D^\omega$, $(E^B_D)^\omega : B^\omega \to D^\omega$ of the conditional expectations E^M_D, E^A_D, E^B_D, respectively.

Proposition 6. *$M' \cap M^\omega = M' \cap D^\omega \subseteq D^\omega$.*

This is a simple application of Proposition 5. To use it, we need the following lemma:

Lemma 7 ([25, Lemma 4.2])**.** *Let $N \supseteq D$ be a pair of a non-type* I *factors with separable predual and a Cartan subalgebra. Let $E^N_D : N \to D$ be the (unique faithful) normal conditional expectation. Then, one can choose a faithful normal state φ on D and a unitary $u \in N_{\varphi\circ E^N_D}$ in such a way that $uDu^* = D$ and $E^N_D(u^n) = 0$ as long as $n \neq 0$.*

Proof of Proposition 6. Thanks to Lemma 7, we can choose a faithful normal state φ on D and a unitary $u \in A_{\varphi\circ E^A_D}$ in such a way that $uDu^* = D$ and $E^A_D(u^n) = 0$ as long as $n \neq 0$. By Proposition 5 ($u = w$ in this case) we have, for $x \in \{u\}' \cap M^\omega$ and for $y \in B^\circ$,

$$\left|\left|y\left(x - (E^M_D)^\omega(x)\right)\right|\right|_{(\varphi\circ E^M_D)^\omega} \leq\ ||yx - xy||_{(\varphi\circ E^M_D)^\omega}\,.$$

Here we do not write the canonical injection. Thanks again to Lemma 7, we can choose the above y to be a unitary in B° so that

$$\begin{aligned} x \in \{u, y\}' \cap M^\omega &\Longrightarrow \left|\left| y\left(x - (E_D^M)^\omega(x)\right)\right|\right|_{(\varphi\circ E_D^M)^\omega} = 0 \\ &\Longrightarrow x = (E_D^M)^\omega(x) \in D^\omega. \end{aligned}$$

Hence we are done. □

As we remarked in §3, the relative commutant $M' \cap M^\omega$ may or may not coincide with the asymptotic centralizer M_ω. Therefore, the following theorem is somewhat nontrivial. The key fact is that the relative commutant $M' \cap M^\omega$ sits in the abelian algebra D^ω.

Theorem 8. *We have*

$$M_\omega = M' \cap M^\omega = M' \cap D^\omega \subseteq D^\omega.$$

Hence, M is not a McDuff factor ([14], [4])*, and, for every centralizing sequence $(m(k))_{k\in\mathbb{N}}$, we have*

$$\sigma\text{-}s^*\text{-}\lim_{n\to\infty}\left(m(k) - E_D^M\left(m(k)\right)\right) = 0.$$

Proof. Since we know that $M_\omega \subseteq M' \cap M^\omega$, it suffices to show that $M' \cap D^\omega \subseteq M_\omega$, thanks to Proposition 6.

Fix a faithful normal state φ on D, and observe that, for $x = \pi((x(k))_{k\in\mathbb{N}}) \in M^\omega$,

$$x \in M_\omega \Longleftrightarrow \begin{cases} \lim_{k\to\omega} ||[x(k), \varphi\circ E_D^M]|| = 0, \\ \lim_{k\to\omega} ||[x(k), m]||_{\varphi\circ E_D^M} = 0 \quad \text{for every } m \in M. \end{cases} \tag{8}$$

Indeed, we have, for $m_1, m_2 \in M$,

$$\begin{aligned} &\left|[m_1, m_2(\varphi\circ E_D^M)](x)\right| = \left|\varphi\circ E_D^M(xm_1m_2) - \varphi\circ E_D^M(m_1xm_2)\right| \\ &= \left|\varphi\circ E_D^M(xm_1m_2) - \varphi\circ E_D^M(xm_2m_1) + \varphi\circ E_D^M(xm_2m_1) - \varphi\circ E_D^M(m_1xm_2)\right| \\ &\le \left|\varphi\circ E_D^M(xm_1m_2) - \varphi\circ E_D^M(xm_2m_1)\right| + \left|[m_1, \varphi\circ E_D^M](xm_2)\right| \\ &\le \left|\varphi\circ E_D^M(x[m_1, m_2])\right| + \left|\left|[m_1, \varphi\circ E_D^M]\right|\right| \cdot ||m_2|| \cdot ||x|| \\ &\le ||x^*||_{\varphi\circ E_D^M} \cdot ||[m_1, m_2]||_{\varphi\circ E_D^M} + \left|\left|[m_1, \varphi\circ E_D^M]\right|\right| \cdot ||m_2|| \cdot ||x|| \\ &\le \Big(||[m_1, m_2]||_{\varphi\circ E_D^M} + \left|\left|[m_1, \varphi\circ E_D^M]\right|\right| \cdot ||m_2||\Big) \cdot ||x||. \end{aligned}$$

Here, the fifth inequality comes from the Cauchy-Schwarz inequality. Hence the right-hand side of (8) implies that $\lim_{k\to\omega}\left|\left|[x(k), m(\varphi\circ E_D^M)]\right|\right| = 0$ for every $m \in M$. Thus, $\lim_{k\to\omega} ||[x(k), \psi]|| = 0$ for every $\psi \in M_*$ since the set of elements $m(\varphi\circ E_D^M)$ $(m \in M)$ is dense in M_*. Since D sits in the centralizer $M_{\varphi\circ E_D^M}$, we have

$$x = \pi\left((x(k))_{k\in\mathbb{N}}\right) \in D^\omega \quad \Longrightarrow \quad \lim_{k\to\omega}\left|\left|[x(k), \varphi\circ E_D^M]\right|\right| = 0.$$

Therefore, we get the desired inclusion relation $M' \cap D^\omega \subseteq M_\omega$ thanks to (8). □

A von Neumann factor (with separable predual) is said to be full if the inner automorphism group is closed in the u-topology ([3]). An equivalent condition is that the asymptotic centralizer is trivial, i.e., 1-dimensional. (See [3, Corollary 3.6].) Thus, Theorem 8 says that our amalgamated free product $M = A *_D B$ is a full factor if and only if $M' \cap D^\omega\,(= M_\omega) = \mathbf{C}1$.

Remark 9. The relative commutant $M' \cap D^\omega$ is trivial if and only if, for a sequence $(p(k))_{k\in\mathbb{N}}$ of projections in D, we have

$$\left(\lim_{k\to\omega} ||p(k) - \mathrm{Ad}\,u\,(p(k))||_\varphi = 0,\ \forall u \in \mathcal{N}_M(D)\right) \Longrightarrow \lim_{k\to\omega} ||p(k)||_\varphi \cdot ||1-p(k)||_\varphi = 0.$$

The proof is as follows:

("only if" part): For each $(x(k))_{k\in\mathbb{N}} \in \mathcal{I}_\omega^M$, we have

$$\begin{aligned} ||x(k)p(k)||^2_{\varphi\circ E_D^M} &= \varphi \circ E_D^M(p(k)x(k)^*x(k)p(k)) \\ &= \varphi \circ E_D^M(p(k)x(x)^*x(k)) \quad (\text{since } p(k) \text{ is in } D \subseteq M_{\varphi\circ E_D^M}) \\ &\leq ||p(k)||_{\varphi\circ E_D^M} \cdot ||x(k)^*x(k)||_{\varphi\circ E_D^M} \to 0 \quad (\text{as } k \to \omega). \end{aligned}$$

Here, the third inequality comes from the Cauchy-Schwarz inequality. In this way, we see that $(p(k))_{k\in\mathbb{N}}$ is in $\mathcal{M}(\mathcal{I}_\omega^M)$. Let $p = \pi((p(k))_{k\in\mathbb{N}}) \in M^\omega$. Then we have

$$||p - upu^*||_{(\varphi\circ E_D^M)^\omega} = \lim_{k\to\omega} ||p(k) - up(k)u^*||_{\varphi\circ E_D^M} = \lim_{k\to\omega} ||p(k) - up(k)u^*||_\varphi = 0$$

for every $u \in \mathcal{N}_M(D)$, and thus we see that $p \in M' \cap D^\omega$, since $\mathcal{N}_M(D)$ generates the whole M. Therefore, the assumption implies that p must be either 0 or 1, that is,

$$\lim_{k\to\omega} ||p(k)||_\varphi \cdot ||1-p(k)||_\varphi = ||p||_{(\varphi\circ E_D^M)^\omega} \cdot ||1-p||_{(\varphi\circ E_D^M)^\omega} = 0.$$

Hence we are done.

("if" part): It suffices to show that each projection in $M' \cap D^\omega$ must be either 0 or 1. Let p be a projection in $M' \cap D^\omega$. Then we may and do assume that p is represented as $p = \pi((p(k))_{k\in\mathbb{N}})$ with projections $p(k)$ in D (see [4, Lemma 1.1.5]). Since p is in $M' \cap D^\omega$, we have

$$\lim_{k\to\omega} ||p(k) - up(k)u^*||_\varphi = ||p - upu^*||_{(\varphi\circ E_D^M)^\omega} = 0$$

for every $u \in \mathcal{N}_M(D)$. Hence the assumption implies that

$$||p||_{(\varphi\circ E_D^M)^\omega} \cdot ||1-p||_{(\varphi\circ E_D^M)^\omega} = \lim_{k\to\omega} ||p(k)||_\varphi \cdot ||1-p(k)||_\varphi = 0,$$

and thus p is 0 or 1.

An ergodic countable nonsingular equivalence relation $\mathcal{R}$ over a (non-atomic) standard Borel probability space (X,μ) is strongly ergodic ([22], [23]) if, for a sequence $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of X, we have

$$\left(\lim_{n\to\infty} \mu\,(B_n \bigtriangleup \phi(B_n)) = 0,\ \forall \phi \in [\mathcal{R}]\right) \Longrightarrow \lim_{n\to\infty} \mu(B_n)\cdot(1-\mu(B_n)) = 0,$$

where $[\mathcal{R}]$ denotes the full group of $\mathcal{R}$, i.e., the set of transformations $X \to X$ whose graphs sit in $\mathcal{R}$. It is plain to see that the latter condition in Remark 9 is equivalent to the strong ergodicity of the canonical equivalence relation $\mathcal{R}_M$, since $\mathcal{N}_M(D)$ is generated by $\mathcal{N}_A(D)$ and $\mathcal{N}_B(D)$. Therefore, we conclude

Corollary 10. *The amalgamated free product $M = A *_D B$ becomes a full factor if and only if the canonical equivalence relation $\mathcal{R}_M$ is strongly ergodic.*

The following corollary is perhaps known to specialists.

Corollary 11 (cf. Connes [3]). *No type* III_0 *ergodic countable nonsingular equivalence relation* $\mathcal{R}$ *is strongly ergodic.*

Proof. We consider the case that $\mathcal{R}_A = \mathcal{R}_B = \mathcal{R}$, and then the canonical equivalence relation $\mathcal{R}_M$ is just $\mathcal{R}$. Since $\mathcal{R}$ is of type III_0, so is the amalgamated free product M (see [26, Remarks 4.8 (1)]). We know (see [3]) that no type III_0 factor (with separable predual) is full. Therefore, $\mathcal{R}$ need not be strongly ergodic. $\square$

From each $c \in Z^1(\mathcal{R}_M, \mathbb{T})$, we get two 1-cocycles $c^A := c|_{\mathcal{R}_A}$, $c^B := c|_{\mathcal{R}_B}$ on $\mathcal{R}_A$, $\mathcal{R}_B$, respectively. We set

$$\alpha_c^A := M_{c^A}^{\sigma_A}, \quad \alpha_c^B := M_{c^B}^{\sigma_B}.$$

Since $\alpha_c^A|_D = \mathrm{Id}$ and $\alpha_c^B|_D = \mathrm{Id}$, we get the group homomorphism

$$\Phi : c \in Z^1(\mathcal{R}_M, \mathbb{T}) \mapsto \Phi(c) := \alpha_c^A *_D \alpha_c^B \in \mathrm{Aut}(M),$$

thanks to [25, Proposition 2.5]. It is plain to see that

$$\Phi(B^1(\mathcal{R}_M, \mathbb{T})) = \mathrm{Int}(M, D) \subseteq \mathrm{Int}(M),$$

and hence

$$\widehat{\Phi} : [c] \in H^1(\mathcal{R}_M, \mathbb{T}) \mapsto \widehat{\Phi}([c]) := \varepsilon(\Phi(c)) \in \mathrm{Out}(M)$$

is a well-defined group homomorphism, where $\varepsilon : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ is the canonical quotient map.

Proposition 12. *The group homomorphism* $\widehat{\Phi}$ *is injective.*

Proof. Suppose that $\varepsilon(\Phi(c)) = \varepsilon(\mathrm{Id})$, i.e., $\Phi(c) = \mathrm{Ad}\, X$ with a unitary $X \in M$. Then it suffices to show that $X = E_D^M(X) \in D$. Lemma 7 enables us to choose a faithful normal state φ on D and unitaries $u \in A_{\varphi\circ E_D^A}$, $v \in B$ such that $uDu^* = D$, $vDv^* = D$ and

$$E_D^A(u^n) = 0, \quad E_D^B(v^n) = 0$$

as long as $n \neq 0$. Then it is plain to see that

$$X^* = uX^*\alpha_c^A(u)^*, \tag{9}$$

$$vX^* = X^*\alpha_c^B(v), \tag{10}$$

$$E_D^A \circ \alpha_c^A = E_D^A \quad \left(E_D^B \circ \alpha_c^B = E_D^B\right). \tag{11}$$

Thanks to (11), we see that $\alpha_c^A(u)$ is also in $A_{\varphi\circ E_D^A}$, and that

$$\alpha_c^A(u)D\alpha_c^A(u)^* = D, \quad E_D^A\left(\left(\alpha_c^A(u)\right)^n\right) = 0 \quad (\text{as long as } n \neq 0).$$

Therefore, we have

$$\begin{aligned} \left|\left|v(X^* - E_D^M(X)^*)\right|\right|_{\varphi\circ E_D^M} &= \left|\left|v\left(X^* - E_D^M(X)^*\right)\right|\right|_{(\varphi\circ E_D^M)^\omega} \\ &\leq \left|\left|vX^* - X^*\alpha_c^B(v)\right|\right|_{(\varphi\circ E_D^M)^\omega} = 0 \end{aligned}$$

thanks to Theorem 4 together with (9) and then (10) and hence $X = E_D^M(X)$ since v is a unitary. $\square$

Here is a simple lemma.

Lemma 13. *Let $A \supseteq D \subseteq B$ be von Neumann algebras, and assume that they are contained in a σ-finite von Neumann algebra N. Suppose that there are faithful normal conditional expectations*

$$E_D^N : N \to D, \quad E_D^A : A \to D, \quad E_D^B : B \to D$$
$$E_A^N : N \to A, \quad E_B^N : N \to B$$

such that

$$E_D^N = E_D^A \circ E_A^N = E_D^B \circ E_B^N.$$

Suppose also that N is generated by A and B, and that D is finite, that is, it has a faithful normal tracial state φ. If a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in D produces the two automorphisms

$$\alpha_A = u\text{-}\lim_{n\to\infty}(\operatorname{Ad} u_n)|_A \in \operatorname{Aut}(A) \quad \textit{and} \quad \alpha_B = u\text{-}\lim_{n\to\infty}(\operatorname{Ad} u_n)|_B \in \operatorname{Aut}(B),$$

then one can construct a unique extension $\widetilde{\alpha} \in \operatorname{Aut}(N)$ of α_A and α_B, i.e.,

$$\widetilde{\alpha} = u\text{-}\lim_{n\to\infty} \operatorname{Ad} u_n. \tag{12}$$

The converse also holds. Namely, if an automorphism on N is as in (12)*, then its restrictions to A and B are approximated by common inner automorphisms of the special form $\operatorname{Ad} u$ with $u \in \mathcal{U}(D)$ in the u-topology, simultaneously.*

Proof. The unitaries u_n are in D, and hence in the centralizer of $\varphi\circ E_D^N$. Therefore, the automorphisms $\operatorname{Ad} u_n$, $n \in \mathbb{N}$, are in $\operatorname{Aut}_{\varphi\circ E_D^N}(N)$, and also $(\operatorname{Ad} u_n)|_A$ and $(\operatorname{Ad} u_n)|_B$, $n \in \mathbb{N}$, are in $\operatorname{Aut}_{\varphi\circ E_D^A}(A)$ and in $\operatorname{Aut}_{\varphi\circ E_D^B}(B)$ respectively. Hence, we may and do assume that

$$p\text{-}\lim_{n\to\infty}(\operatorname{Ad} u_n)|_A \in \operatorname{Aut}(A) \quad \text{and} \quad p\text{-}\lim_{n\to\infty}(\operatorname{Ad} u_n)|_B \in \operatorname{Aut}(B)$$

(see §4), and it suffices to show that

$$p\text{-}\lim_{n\to\infty} \operatorname{Ad} u_n = \widetilde{\alpha}$$

for some automorphism $\widetilde{\alpha} \in \operatorname{Aut}_{\varphi\circ E_D^N}(N)$. For every monomial x in $*$-alg$(A \cup B)$, the sequence $((\operatorname{Ad} u_n)(x))_{n\in\mathbb{N}}$ converges in the strong operator topology, since the product operation is strong operator continuous on each bounded ball, and hence we can define an isometry $U : L^2(N, \varphi\circ E_D^N) \to L^2(N, \varphi\circ E_D^N)$ in such a way that

$$U\Lambda_{\varphi\circ E_D^N}(x) := \lim_{n\to\infty} \Lambda_{\varphi\circ E_D^N}((\operatorname{Ad} u_n)(x))$$

for each $x \in N$, because $\varphi\circ E_D^N \circ (\operatorname{Ad} u_n) = \varphi\circ E_D^N$ for all $n \in \mathbb{N}$. Here we used the notation rule given before Proposition 5. By the same reasoning, we can also define the isometry $V : L^2(N, \varphi\circ E_D^N) \to L^2(N, \varphi\circ E_D^N)$ by

$$V\Lambda_{\varphi\circ E_D^N}(x) := \lim_{n\to\infty} \Lambda_{\varphi\circ E_D^N}((\operatorname{Ad} u_n^*)(x))$$

for each $x \in N$. Then we have, for $x, y \in N$,

$$\begin{aligned}\left(U^*\Lambda_{\varphi\circ E_D^N}(x)\middle|\, \Lambda_{\varphi\circ E_D^N}(y)\right)_{\varphi\circ E_D^N} &= \left(\Lambda_{\varphi\circ E_D^N}(x)\middle|\, U\Lambda_{\varphi\circ E_D^N}(y)\right)_{\varphi\circ E_D^N}\\ &= \lim_{n\to\infty}\left(\Lambda_{\varphi\circ E_D^N}(x)\middle|\, \Lambda_{\varphi\circ E_D^N}\left((\mathrm{Ad}\, u_n)(y)\right)\right)_{\varphi\circ E_D^N}\\ &= \lim_{n\to\infty}\left(\Lambda_{\varphi\circ E_D^N}\left((\mathrm{Ad}\, u_n^*)(x)\right)\middle|\, \Lambda_{\varphi\circ E_D^N}(y)\right)_{\varphi\circ E_D^N}\\ &= \left(V\Lambda_{\varphi\circ E_D^N}(x)\middle|\, \Lambda_{\varphi\circ E_D^N}(y)\right)_{\varphi\circ E_D^N},\end{aligned}$$

and hence $U^* = V$, an isometry, so that U is a unitary.

It is easy to see that

$$\mathrm{Ad}\, U(x) = s.o.\text{-}\lim_{n\to\infty}(\mathrm{Ad}\, u_n)(x) \quad \text{on } L^2(N, \varphi\circ E_D^N)$$

for every $x \in N$, and hence $\mathrm{Ad}\, U$ defines a desired automorphism $\widetilde{\alpha} \in \mathrm{Aut}_{\varphi\circ E_D^N}(N)$.

The converse is clear from what we mentioned at the beginning of this proof. □

Theorem 14. *We have*

$$\widehat{\Phi}\left(\overline{B^1(\mathcal{R}_M,\mathbb{T})}/B^1(\mathcal{R}_M,\mathbb{T})\right) = \varepsilon\left(\Phi\left(\overline{B^1(\mathcal{R}_M,\mathbb{T})}\right)\right) = \varepsilon\left(\overline{\mathrm{Int}(M)}\right),$$

and hence, via the injective group homomorphism $\widehat{\Phi}$,

$$\chi(M) = \overline{\mathrm{Int}(M)}/\mathrm{Int}(M) \cong \overline{B^1(\mathcal{R}_M,\mathbb{T})}/B^1(\mathcal{R}_M,\mathbb{T}) \;\left(\subseteq H^1(\mathcal{R}_M,\mathbb{T})\right).$$

Proof. Let us choose an automorphism β from $\overline{\mathrm{Int}(M)}$. By Proposition 2 and Theorem 8, there are a unitary $X \in M$ and an automorphism $\beta_0 \in \mathrm{Aut}(M, D)$ satisfying

$$\beta = (\mathrm{Ad}\, X)\circ\beta_0, \quad \beta_0 = u\text{-}\lim_{n\to\infty}(\mathrm{Ad}\, u_n) \text{ with } u_n \in \mathcal{U}(D).$$

Then we get two automorphisms in $\mathrm{Aut}(A, D)$, $\mathrm{Aut}(B, D)$, respectively,

$$\beta_0^A := \beta_0|_A = u\text{-}\lim_{n\to\infty}(\mathrm{Ad}\, u_n)|_A, \quad \beta_0^B := \beta_0|_B = u\text{-}\lim_{n\to\infty}(\mathrm{Ad}\, u_n)|_B,$$

thanks to the latter assertion of Lemma 13, and it is clear that

$$\beta_0 = \beta_0^A *_D \beta_0^B.$$

Via the point realization $D = L^\infty(X)$, the unitary u_n can be regarded as a measurable function on X taking values in $\mathbb{T}$, and we define two measurable functions c_n^A, c_n^B on $\mathcal{R}_A$, $\mathcal{R}_B$, respectively, as follows:

$$c_n^A(x,y) := u_n(x)u_n(y)^{-1}, \quad c_n^B(x,y) := u_n(x)u_n(y)^{-1}.$$

Then we have

$$(\mathrm{Ad}\, u_n)|_A = M_{c_n^A}^{\sigma_A}, \quad (\mathrm{Ad}\, u_n)|_B = M_{c_n^B}^{\sigma_B},$$

and c_n^A and c_n^B are coboundaries. By Proposition 1 we can write

$$\beta_0^A = M_{c^A}^{\sigma_A}, \quad \beta_0^B = M_{c^B}^{\sigma_B}$$

with relevant 1-cocycles c^A, c^B, and get

$$c_n^A \to c^A, \quad c_n^B \to c^B$$

in probability. We then consider

$$W^*(\mathcal{R}_M) = W^*(\mathcal{R}_A)\vee W^*(\mathcal{R}_B) \supseteq L^\infty(X).$$

The 1-cocycles c_n^A, c^A, c_n^B, c^B produce the automorphisms

$$M_{c_n^A} = (\operatorname{Ad} u_n)|_A,\ M_{c^A} \in \operatorname{Aut}(W^*(\mathcal{R}_A), L^\infty(X)),$$
$$M_{c_n^B} = (\operatorname{Ad} u_n)|_B,\ M_{c^B} \in \operatorname{Aut}(W^*(\mathcal{R}_B), L^\infty(X)),$$

respectively, and Proposition 1 implies that

$$u\text{-}\lim_{n\to\infty} M_{c_n^A} = M_{c^A}, \quad u\text{-}\lim_{n\to\infty} M_{c_n^B} = M_{c^B}.$$

Then by the first assertion of Lemma 13 together with Proposition 1, we get a 1-cocycle $c \in Z^1(\mathcal{R}_M, \mathbb{T})$ satisfying

$$M_c = u\text{-}\lim_{n\to\infty} (\operatorname{Ad} u_n) \in \overline{\operatorname{Int}(W^*(\mathcal{R}_M), L^\infty(X))},$$

so that the 1-cocycle c is in $\overline{B^1(\mathcal{R}_M, \mathbb{T})}$. Since

$$M_c|_{W^*(\mathcal{R}_A)} = u\text{-}\lim_{n\to\infty} (\operatorname{Ad} u_n)|_{W^*(\mathcal{R}_A)} = M_{c^A},$$
$$M_c|_{W^*(\mathcal{R}_B)} = u\text{-}\lim_{n\to\infty} (\operatorname{Ad} u_n)|_{W^*(\mathcal{R}_B)} = M_{c^B},$$

we see that $c|_{\mathcal{R}_A} = c^A$, $c|_{\mathcal{R}_B} = c^B$ and that

$$\beta_0^A = M_{c^A}^{\sigma_A} = \alpha_c^A, \quad \beta_0^B = M_{c^B}^{\sigma_B} = \alpha_c^B.$$

Then we have

$$\varepsilon(\Phi(c)) = \varepsilon(\alpha_c^A *_D \alpha_c^B) = \varepsilon(\beta_0^A *_D \beta_0^B) = \varepsilon(\beta_0) = \varepsilon(\beta).$$

Therefore, we get the inclusion relation

$$\varepsilon\left(\overline{\operatorname{Int}(M)}\right) \subseteq \varepsilon\left(\Phi\left(\overline{B^1(\mathcal{R}_M, \mathbb{T})}\right)\right).$$

Let us choose a 1-cocycle c from $\overline{B^1(\mathcal{R}_M, \mathbb{T})}$, and then we can choose a sequence $(c_n)_{n\in\mathbb{N}}$ of coboundaries with $c_n \to c$ in probability. Then we have

$$\alpha_{c_n}^A = M_{c_n^A}^{\sigma_A} = (\operatorname{Ad} u_n)\,|_A \in \operatorname{Int}(A, D), \quad \alpha_{c_n}^B = M_{c_n^B}^{\sigma_B} = (\operatorname{Ad} u_n)\,|_B \in \operatorname{Int}(B, D),$$

where $c_n(x, y) = u_n(x)u_n(y)^{-1}$ with a measurable function $u_n : X \to \mathbb{T}$. We here identify, as in §2, the function u_n with an operator in $D = L^\infty(X)$. Proposition 1 enables us to see that

$$\alpha_c^A = u\text{-}\lim_{n\to\infty} \alpha_{c_n}^A = u\text{-}\lim_{n\to\infty} (\operatorname{Ad} u_n)\,|_A, \quad \alpha_c^B = u\text{-}\lim_{n\to\infty} \alpha_{c_n}^B = u\text{-}\lim_{n\to\infty} (\operatorname{Ad} u_n)\,|_B,$$

and hence the first assertion of Lemma 13 implies that

$$\Phi(c) = \alpha_c^A *_D \alpha_c^B = u\text{-}\lim_{n\to\infty} (\operatorname{Ad} u_n) \in \overline{\operatorname{Int}(M, D)} \subseteq \overline{\operatorname{Int}(M)}.$$

Therefore, we get the reverse inclusion relation

$$\varepsilon\left(\Phi\left(\overline{B^1(\mathcal{R}_M, \mathbb{T})}\right)\right) \subseteq \varepsilon\left(\overline{\operatorname{Int}(M)}\right).$$

Hence we have proved the first assertion.

Although the latter assertion follows from one defintion of the χ-group $\chi(M)$, i.e., $\chi(M) =$ the center $\left(\overline{\operatorname{Int}(M)}/\operatorname{Int}(M)\right)$, and what we have shown here, we do give the following simple argument based on the "other" definition: Since $\Phi(c)|_D = \mathrm{Id}$ for every $c \in Z^1(\mathcal{R}_M, \mathbb{T})$, we have $\Phi\left(Z^1(\mathcal{R}_M, \mathbb{T})\right) \subseteq \operatorname{Ct}(M)$ thanks to Theorem 8, and hence

$$\overline{\operatorname{Int}(M)} = \operatorname{Int}(M) \cdot \Phi\left(\overline{B^1(\mathcal{R}_M, \mathbb{T})}\right) \subseteq \operatorname{Ct}(M)$$

by the discussion above. Hence we get the latter assertion of Theorem 14. $\square$

Corollary 15. *If the canonical equivalence relation $\mathcal{R}_M$ is amenable (or hyperfinite), then we have*
$$\chi(M) \cong H^1(\mathcal{R}_M, \mathbb{T}) \neq \{\mathrm{Id}\}.$$

Proof. This is a simple consequence of Theorem 14 and a result in ergodic theory. In their paper [19] (also see [23]), Parthasarathy and Schmidt showed that, if $\mathcal{R}$ is an ergodic hyperfinite (= amenable) countable nonsingular equivalence relation over a non-atomic standard Borel probability space, then the coboundaries $B^1(\mathcal{R}, \mathbb{T})$ is dense in $Z^1(\mathcal{R}, \mathbb{T})$ in the topology of convergence in probability, but $B^1(\mathcal{R}, \mathbb{T}) \neq Z^1(\mathcal{R}, \mathbb{T})$. This fact and Theorem 14 immediately imply the assertion. □

The next is the opposite situation.

Corollary 16. *The χ-group $\chi(M)$ is trivial if and only if the canonical equivalence relation $\mathcal{R}_M$ is strongly ergodic.*

Proof. This immediately follows from Theorem 14 and Corollary 10. □

Remarks 17. A few remarks are in order.

(1) In his paper [13], Mackey showed that, for any separable topological group G with subgroup H, the induced Borel structure of G/H is countably separated if and only if H is closed. Hence, the χ-group $\chi(M)$ is either not countably separated or the trivial group, and unfortunately the χ-group $\chi(M)$ is not enough to understand our amalgamated free product M.

(2) Summing up the results in this section, we see that the following conditions are equivalent.

- The amalgamated free product $M = A *_D B$ is a full factor.
- The χ-group $\chi(M)$ is trivial.
- The canonical equivalence relation $\mathcal{R}_M$ is strongly ergodic.

(3) We would like to mention that the result [21, Proposition 2.3] of Schmidt follows from its von Neumann factor version [3, Corollary 3.6] via Theorem 14. Let $\mathcal{R}$ be an ergodic countable nonsingular Borel equivalence relation over a non-atomic standard Borel probability space (X, μ), and set
$$N := W^*(\mathcal{R}) *_{L^\infty(X)} W^*(\mathcal{R}).$$
The canonical equivalence relation associated with N is just $\mathcal{R}$, and hence Corollary 10 says that $\mathcal{R}$ is strongly ergodic if and only if N is a full factor, i.e., $\mathrm{Int}(N)$ is closed. (We are making use of [3, Corollary 3.6] just here.) By Theorem 14, we have
$$\overline{\mathrm{Int}(N)}/\mathrm{Int}(N) = \chi(N) = \overline{B^1(\mathcal{R}, \mathbb{T})}/B^1(\mathcal{R}, \mathbb{T}),$$
and hence $\mathrm{Int}(N)$ is closed if and only if $B^1(\mathcal{R}, \mathbb{T})$ is closed. Hence we are done.

Example 18. ([25, Example 4.6]) Let
$$X = \prod_{n=1}^{\infty} \{0, 1\}, \quad d\mu_\lambda = \prod_{n=1}^{\infty} \left\{ \frac{1}{1+\lambda^{1/2}}, \frac{\lambda^{1/2}}{1+\lambda^{1/2}} \right\},$$
and let $\mathfrak{S}_\infty$ be the group of finite permutations on the product components of the infinite product space X. We define the automorphism θ on (X, μ_λ) by
$$(\theta(x))_k = \begin{cases} x_k & \text{if } k \geq 2, \\ x_k + 1 \pmod 2 & \text{if } k = 1, \end{cases}$$

for $x = (x_k)_{k\in\mathbb{N}} \in X$. Let $\mathcal{R}_A$ and $\mathcal{R}_B$ be the countable nonsingular Borel equivalence relations over (X, μ_λ) constructed from the groups $\mathfrak{S}_\infty$ and $\theta\mathfrak{S}_\infty\theta := \{\theta \circ \sigma \circ \theta\ ;\ \sigma \in \mathfrak{S}_\infty\}$, respectively. The probability measure μ_λ is invariant under $\mathcal{R}_A$, while $\mathcal{R}_B$ preserves the other probability measure $\mu_\lambda \circ \theta$, as is easily seen. It is known that both $\mathcal{R}_A$ and $\mathcal{R}_B$ are ergodic and amenable. We consider the Feldman-Moore factors

$$A := W^*(\mathcal{R}_A), \quad B := W^*(\mathcal{R}_B),$$

which are both isomorphic to the hyperfinite type II_1 factor and contain a common Cartan subalgebra $D := L^\infty(X, \mu_\lambda)$. Then the amalgamated free product $M = A *_D B$ is shown to be a factor of type III_λ. (The details can be found in [25, Example 4.6].) Since the canonical equivalence relation $\mathcal{R}_M$ is a subrelation of the amenable type $\mathrm{III}_{\lambda^{1/2}}$ equivalence relation constructed from the odometer action on (X, μ_λ), the canonical equivalence relation $\mathcal{R}_M$ is amenable, and hence Corollary 10 says that the amalgamated free product M is not a full factor, not a McDuff factor, and by Theorem 14,

$$\chi(M) = H^1(\mathcal{R}_M, \mathbb{T}) \neq \{\mathrm{Id}\}.$$

Such a type III_1 example can also be constructed in a similar fashion, while there is no such type III_0 factor.

Example and Remark 19. The factor $N = L^\infty(X) \rtimes_\sigma \mathbb{F}_2$ constructed from the so-called Bernoulli shift σ of $\mathbb{F}_2 = \langle a, b\rangle$ on the infinite product measure space

$$X = \prod_{g\in\mathbb{F}_2} \{0, 1\}, \quad d\mu = \prod_{g\in\mathbb{F}_2} \left\{\frac{1}{2}, \frac{1}{2}\right\}$$

can be regarded as an example of an amalgamated free product of two hyperfinite type II_1 factors over a common Cartan subalgebra, since σ_a, σ_b are both ergodic. The factor N is known to be full (e.g. [3, Proposition 3.9]), and as a simple consequence of our main result, we see that this factor is not isomorphic to the amalgamated free product (over a common Cartan subalgebra) $R *_D R$ with the same $R \supseteq D$, a pair consisting of the hyperfinite type II_1 factor and a Cartan subalgebra. (Note that $R *_D R$ is independent from the choice of D in R, thanks to the Connes-Feldman-Weiss result [7].) Related to this fact, we would like here to ask the following question: *Does the amalgamated free product $R *_D R$ have a Cartan subalgebra?* We should make one simple comment: The subalgebra D is not a Cartan subalgebra (or more precicely, not maximal abelian) in $R *_D R$, as was shown before (see the beginning of this section).

To conclude this section, we would like to discuss a relation with work of Kosaki [12]. As was mentioned before, the triple $A \supseteq D \subseteq B$ produces two countable nonsingular Borel equivalence relations $\mathcal{R}_A$, $\mathcal{R}_B$ over a standard Borel probability space (X, μ). Thus, it is quite natural to imagine that the amalgamated free product $M = A *_D B$ can be captured as the convolution von Neumann algebra associated with the "free product groupoid" of the equivalence relations $\mathcal{R}_A$, $\mathcal{R}_B$ over the unit measure space (X, μ). Indeed, this is true; that is, Kosaki gave a construction of such a free product in a rigorous measurable fashion and identified M with the convolution von Neumann algebra. In general, the free product groupoid $\Gamma = \mathcal{R}_A *_X \mathcal{R}_B$ is not an equivalence relation but a groupoid, so that one may consider its associated equivalence relation

$$\mathcal{R}_\Gamma := \{(r(\gamma), s(\gamma)) \in X^2\ ;\ \gamma \in \Gamma\}$$

with the range/source maps $r, s : \Gamma \to X$.

The equivalence relation $\mathcal{R}_\Gamma$ can easily be checked to coincide with the canonical equivalence relation $\mathcal{R}_M$. It is known that any groupoid can be captured in principle from its associated equivalence relation and its isotropy bundle. (See e.g. [24].) Therefore, it is somewhat mysterious that only the associated equivalence relation $\mathcal{R}_\Gamma$ (or the canonical equivalence relation $\mathcal{R}_M$) determines whether or not the convolution von Neumann algebra is full in our case, and hence we should ask about the reason and investigate the isotropy bundle in detail. Under an extra (but quite natural) assumption, we can show that almost every group appearing in the isotropy bundle $\bigsqcup_{x\in X} \Gamma_x^x$ with $\Gamma_x^x := \{\gamma \in \Gamma\ ;\ r(\gamma) = s(\gamma) = x\}$ is of Haagerup type ([11]). The details will be discussed elsewhere.

5. Appendix

Let $A \supseteq D \subseteq B$ be von Neumann algebras with separable preduals and a common Cartan subalgebra. After fixing the point realization $D = L^\infty(X)$ with a standard Borel probability space (X, μ), we can construct a unique pair $\mathcal{R}_A$, $\mathcal{R}_B$ of countable nonsingular Borel equivalence relations over X in such a way that

$$A = W^*_{\sigma_A}(\mathcal{R}_A), \quad B = W^*_{\sigma_B}(\mathcal{R}_B)$$

with relevant 2-cocycles σ_A, σ_B. Here the uniqueness means precisely in the isomorphic sense; that is, if $\mathcal{R}'_A$, $\mathcal{R}'_B$ is another such pair over another Y, then there is a measurable isomorphism $\phi : X \to Y$ such that

$$(\phi \times \phi)(\mathcal{R}_A) = \mathcal{R}'_A, \quad (\phi \times \phi)(\mathcal{R}_B) = \mathcal{R}'_B.$$

Indeed, we can prove that, by the straightforward adaptation of the method of Feldman and Moore [9, II]. However, we give here a detailed account, since the canonical equivalence relation

$$\mathcal{R}_M = \mathcal{R}_A \vee \mathcal{R}_B \subseteq X^2$$

plays a crucial role in our analysis.

Let $(A, \mathcal{H}_A, J_A, \mathcal{P}^\natural_A)$ and $(B, \mathcal{H}_B, J_B, \mathcal{P}^\natural_B)$ be the standard forms associated with A and B. Then, following [9, II], we consider the abelian von Neumann algebras

$$C_A := D \vee J_A D J_A \text{ on } \mathcal{H}_A, \quad C_B := D \vee J_B D J_B \text{ on } \mathcal{H}_B,$$

which produce two standard Borel spaces $\mathcal{R}_A$, $\mathcal{R}_B$ (which will be shown later to be equivalence relations up to null sets) with

$$C_A = L^\infty(\mathcal{R}_A), \quad C_B = L^\infty(\mathcal{R}_B).$$

Then the measurable mappings

$$\pi_\ell^{\mathcal{R}_A}, \pi_r^{\mathcal{R}_A} : \mathcal{R}_A \to X, \quad \pi_\ell^{\mathcal{R}_B}, \pi_r^{\mathcal{R}_B} : \mathcal{R}_B \to X$$

are constructed in such a way that

$(\pi_\ell^{\mathcal{R}_A})^* =$ the natural embedding ι_A^X of $D = L^\infty(X)$ into $C_A = L^\infty(\mathcal{R}_A)$,
$(\pi_r^{\mathcal{R}_A})^* =$ the isomorphism $j_A^X : d \in D = L^\infty(X) \mapsto J_A d^* J_A \in C_A = L^\infty(\mathcal{R}_A)$,
$(\pi_\ell^{\mathcal{R}_B})^* =$ the natural embedding ι_B^X of $D = L^\infty(X)$ into $C_B = L^\infty(\mathcal{R}_B)$,
$(\pi_r^{\mathcal{R}_B})^* =$ the isomorphism $j_B^X : d \in D = L^\infty(X) \mapsto J_B d^* J_B \in C_B = L^\infty(\mathcal{R}_B)$.

Via these mappings, the standard Borel spaces $\mathcal{R}_A$, $\mathcal{R}_B$ can indeed be realized as measurable subsets of X^2. Therefore, the arguments in [9, II, pp. 336–348] work for both $A \supseteq D = L^\infty(X)$ and $B \supseteq D = L^\infty(X)$ over the same space X simultaneously, and hence $\mathcal{R}_A$, $\mathcal{R}_B$ are shown to be equivalence relations over X (up to null sets) satisfying

$$A = W^*_{\sigma_A}(\mathcal{R}_A), \quad B = W^*_{\sigma_B}(\mathcal{R}_B)$$

with relevant 2-cocycles.

The uniqueness of the pair $\mathcal{R}_A$, $\mathcal{R}_B$ can also be seen as follows. Let $\mathcal{R}'_A$, $\mathcal{R}'_B$ be another such pair over another Y. Then there are two surjective $*$-isomorphisms

$$\Phi_A : W^*_{\sigma'_A}(\mathcal{R}'_A) \to W^*_{\sigma_A}(\mathcal{R}_A), \quad \Phi_B : W^*_{\sigma'_B}(\mathcal{R}'_B) \to W^*_{\sigma_B}(\mathcal{R}_B)$$

satisfying

$$\Phi := \Phi_A|_{L^\infty(Y)} = \Phi_B|_{L^\infty(Y)} : L^\infty(Y) \cong L^\infty(X).$$

The isomorphism Φ gives rise to a measurable isomorphism $\phi : X \to Y$ with $\phi^* = \Phi$, i.e.,

$$\Phi(f)(x) := f(\phi(x)), \quad x \in X,\ f \in L^\infty(X).$$

By the unitary equivalence between standard forms ([10, Theorem 2.3]), we can define the surjective $*$-isomorphisms

$$\widetilde{\Phi}_A : L^\infty(\mathcal{R}'_A) \to L^\infty(\mathcal{R}_A), \quad \widetilde{\Phi}_B : L^\infty(\mathcal{R}'_B) \to L^\infty(\mathcal{R}_B)$$

in such a way that

$$\widetilde{\Phi}_A \circ \iota_A^Y = \iota_A^X \circ \Phi, \quad \widetilde{\Phi}_A \circ j_A^Y = j_A^X \circ \Phi;$$
$$\widetilde{\Phi}_B \circ \iota_B^Y = \iota_B^X \circ \Phi, \quad \widetilde{\Phi}_A \circ j_B^Y = j_B^X \circ \Phi.$$

(See [9, II, p. 348] for details.) Let $\theta_A := (\widetilde{\Phi}_A)_*$, $\theta_B := (\widetilde{\Phi}_B)_*$, that is, measurable maps $\theta_A : \mathcal{R}_A \to \mathcal{R}'_A$, $\theta_B : \mathcal{R}_B \to \mathcal{R}'_B$ satisfying

$$\widetilde{\Phi}_A(f)(x,y) = f(\theta_A(x,y)), \quad \widetilde{\Phi}_B(f)(x,y) = f(\theta_B(x,y)).$$

We have, for $f \in L^\infty(Y)$,

$$\begin{aligned} f(\pi_\ell^{\mathcal{R}'_A}(\theta_A(x,y))) &= \iota_A^Y(f)(\theta_A(x,y)) \\ &= \widetilde{\Phi}_A(\iota_A^Y(f))(x,y) \\ &= \iota_A^X(\Phi(f))(x,y) \\ &= \Phi(f)(x) = f(\phi(x)), \end{aligned}$$

which implies $\pi_\ell^{\mathcal{R}'_A}(\theta_A(x,y)) = \phi(x)$. Similarly, we have

$$\pi_r^{\mathcal{R}'_A}(\theta_A(x,y)) = \phi(y), \quad \pi_\ell^{\mathcal{R}'_B}(\theta_B(x,y)) = \phi(x), \quad \pi_r^{\mathcal{R}'_B}(\theta_B(x,y)) = \phi(y).$$

Hence,

$$\theta_A(x,y) = (\phi(x),\phi(y)), \quad \theta_B(x,y) = (\phi(x),\phi(y)).$$

Therefore, we get

$$(\phi \times \phi)(\mathcal{R}_A) = \mathcal{R}'_A, \quad (\phi \times \phi)(\mathcal{R}_B) = \mathcal{R}'_B.$$

Hence we are done.

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